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Note

Counterexample to a conjecture of Györi on C_{2l}-free bipartite graphs☆

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Abstract

A counterexample on a conjecture of Györi related with C_{2l}-free bipartite graphs is described.

Keywords: Cycles in bipartite graphs; Forbidden subgraphs

In 1997 Györi [2] studied the structure of C_6-free bipartite graphs and the relationship between this problem and some interesting results of Erdös et al. [1] on a number-theoretic problem. Namely, Györi proved a conjecture of Erdös et al. [1] regarding the maximum number of edges that a C_6-free bipartite graph can have. Moreover, he proved another theorem that generalizes the previous one for cycles of longer length. In this paper Györi stated a conjecture [2, p. 373] that apparently contains a misprint[1] and it should have been expressed in this way:

Conjecture 1. If $G=(X,Y)$ is a bipartite graph with color classes X, Y where $|X|=m$, $|Y|=n$, $m^2 \leqslant n$, $3 \leqslant l \leqslant m$ and G has at least $(l-1)n+m-l+2$ edges, then G must contain a cycle of length $2l$.

In a recent paper [3], the same author disproves Conjecture 1 for $l=3$, but leaves the proof or refutation for $l \geqslant 4$ as an open problem. In this note we provide a counterexample that disproves Conjecture 1 when $m \geqslant 2l-1$. Let us denote by $K_{(m,n)}$ the complete bipartite graph with m vertices in the first class and n vertices in the second one. Let us also denote by $d_G(v)$ the degree of the vertex v in the graph G.

Let l, m be integers such that $3 \leqslant l \leqslant 2(l-1) \leqslant m$. We consider the graphs $G_1 = K_{(m-l+1,l-1)}$ and $G_2 = K_{(l-1,n-l+2)}$. Take two vertices $u \in V(G_1)$ and $v \in V(G_2)$ with $d_{G_1}(u) = m-l+1$ and $d_{G_2}(v) = l-1$, and let G be the bipartite

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[1] The original conjectured value $(l-1)n+m-l+1$ (see [2]) may be easily disproved by means of a graph roughly outlined by the author. We appreciate the referee's comments enlightening this misprint.

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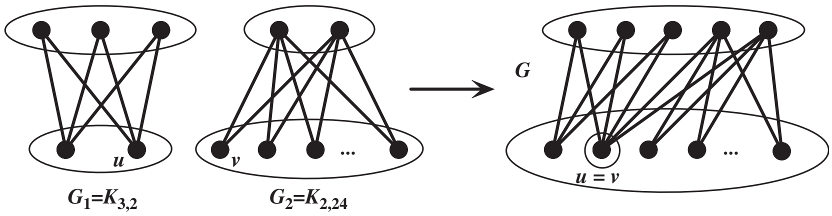

Fig. 1. The graph G with $l = 3$, $m = 5$ and $n = 25$.

graph on m and n vertices obtained by gluing the graphs G_1 and G_2 in such a way that the vertex u of G_1 is identified with the vertex v of G_2 (see Fig. 1).

Clearly, any cycle of G must be entirely contained in either G_1 or G_2. But G_i, $i = 1, 2$, cannot contain a cycle of length $2l$ because one of its classes has cardinality $l - 1$. So G is free of C_{2l} and it has size $e(G) = (m - l + 1)(l - 1) + (l - 1)(n - l + 2) \geqslant (l - 1)n + m - l + 2$ because $m \geqslant 2l - 1$. Therefore, Conjecture 1 is disproved for $m \geqslant 2l - 1$.

In [2], Györi proved the following result:

Theorem. *If $G(X, Y)$ is a bipartite graph with color classes X, Y such that $|X| = m$, $|Y| = n$, $m^2 \leqslant n$ and G has at least $(l - 1)n + c(l)m^2$ edges for some constant $c(l)$ then G must contain a cycle of length $2l$.*

Thus, we propose the following reformulation of the conjecture:

Conjecture 2. If $G = (X, Y)$ is a bipartite graph with color classes X, Y where $|X| = m$, $|Y| = n$, $m^2 \leqslant n$, $3 \leqslant l \leqslant m$ such that $m > (l - 1)^2$ and G has at least $(l - 1)n + 1/(l - 1)m^2$ edges then G must contain a cycle of length $2l$.

Corollary of Theorem 1 in [3] confirms our Conjecture 2 for $l = 3$. For $l \geqslant 4$ it is still an open problem.

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